

Diversity of scales makes an advantage: the case of the Minority Game

Miroslav Pištěk^{a,*}, František Slanina^b

^a*The Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic,
Pod Vodárenskou věží 4, CZ-18208 Praha*

^b*Institute of Physics, Academy of Sciences of the Czech Republic,
Na Slovance 2, CZ-18221 Praha*

Abstract

We use the Minority Game as a testing frame for the problem of the emergence of diversity in socio-economic systems. For the MG with heterogeneous impacts, we show that the direct generalization of the usual agents' profit does not fit some real-world situations. As a typical example we use the traffic formulation of the MG. Taking into account vehicles of various lengths it can easily happen that one of the roads is crowded by a few long trucks and the other contains more drivers but still is less covered by vehicles. Most drivers are in the shorter queue, so the majority win. To describe such situations, we generalized the formula for agents' profit by explicitly introducing utility function depending on an agent's impact. Then, the overall profit of the system may become positive depending on the actual choice of the utility function. We investigated several choices of the utility function and showed that this variant of the MG may turn into a positive sum game.

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1. Introduction

The Minority Game (MG), introduced slightly more than a decade ago [1, 2, 3, 4, 5], gained the reputation of an “Ising model” in the field of Econophysics [6, 7]. It is surely an exaggeration, as the complexity of the Ising model is many orders of magnitude higher, but nevertheless, the MG is a widely accepted framework for testing a variety of ideas in modelling socio-economic phenomena. There are several excellent reviews [8, 9, 10, 11] of the vast literature on the subject, which we have no intention to review again here. The aim of our work is to contribute to understanding the possible advantages of diversity, which may be one of the sources of the heterogeneity of agents.

*Corresponding author

Email addresses: miroslav.pistek@gmail.com (Miroslav Pištěk), slanina@fzu.cz (František Slanina)

The topic of our work falls within the generalisations of the canonical MG to the case of several different groups of agents. Perhaps the most deeply studied is the variant with two types of agents, called producers and speculators [12, 13, 14]. The market ecology which emerges, shows fairly clearly that the two groups need each other to gain better efficiency. In different perspective, the agents may be divided into groups of leaders and imitators [15, 16, 17, 18]. Here also, the performance of the system as a whole might be increased, if the imitators are present in proper fraction. If the agents are allowed to participate in the game or not, depending on their score [19, 20, 21, 22], the bursts of activity reminiscent of the effect of volatility clustering are observed. Similar case considers agents who play the game at certain fixed frequency. Their frequencies may be broadly distributed, which induces new features into the game [23, 24, 25]. The case of investors with different weights was investigated for a very general setup [26]. In our work, we choose a more specific situation and show that it leads to new non-trivial consequences.

From the following consideration it can be seen why can be the diversity helpful in the framework of the MG. In the traffic formulation of the MG, the drivers choose between two alternative roads and those who take the less crowded one, are the winners. If all the vehicles have the same length, only minority of agents can be counted winners, namely those who drive on the road with minority of drivers. However, imagine that besides trucks, there are also small city cars. It can easily happen that one of the roads is crowded by a few long trucks and the other contains more drivers but still is less covered by vehicles. Most drivers are in the shorter queue, so the majority win. This is the essence of our paper which is just mathematical elaboration of this consideration.

However, the previous simple setting does not take into account the utility of the vehicles. In this view, the drivers could choose infinitesimally short vehicles as optimal which clearly contradicts the real traffic situation. Indeed, the longer the car is, the more passengers or load it can carry, obviously for the price of higher fuel consumption. To model these considerations, we incorporated the utility of the vehicles expressed as a function of their length. This way we included feature which favour vehicles of finite length. Nevertheless, the ultimate motivation for the introduction of the utility function is the optimization problem the agents face. We consider the impacts as parameters the agents can choose to maximize their expected utility.

2. Heterogeneous groups in the MG

Generally, the MG is a model of the bounded sources allocation which is driven by agents with bounded rationality. In the simplest setting, they do not have but two options. In each round, they aim to choose the right action, i.e. the action chosen by the minority of agents. Then, only the minority of agents can win for principal reasons. To adapt to their environment, each agent has two strategies. They predict the next optimal choice considering only a few previous results. Still, there is not any space for an adaptation of agents as both their strategies are fixed. Therefore, each agent counts the score of each strategy and then follow the currently better strategy.

In the urban traffic, drivers have cars of various lengths. Following this paradigm, we assign each agent an individual impact on the game. It can be directly interpreted as the length of the vehicle the agent is driving, however, some more ambitious interpretations are left to the conclusion. Next, we divide the agents into several groups according the

assigned impacts. Formally, we have $G \in \mathbb{N}^+$ groups of agents. The group $g \in \{1, \dots, G\}$ contains N_g agents and each of them has impact $I_g \in \mathbb{R}^+$. The total number of agents is $N = \sum_{g=1}^G N_g$ and the relative size of group g will be denoted $\lambda_g \equiv N_g/N$. For further convenience, we introduce the following symbol for the weighted average of an arbitrary function $v_g \in \mathbb{R}$ over all groups

$$[v_g] \equiv \sum_{g=1}^G \lambda_g v_g . \quad (1)$$

Throughout this paper, we shall refer to an actual values of the groups' ratios $\{\lambda_g\}_g$ and impacts $\{I_g\}_g$ as the configuration of the game. To restore the canonical MG, it suffices to set G , λ_1 and I_1 equal to 1.

As usual in the MG, all agents are subject to the same external information and all of them are able to distinguish P different information (or memory) patterns $\mu \in \{1, \dots, P\}$. These patterns arrive randomly with uniform probability. We stress that throughout this paper we shall work within batch MG with uniformly distributed random memories, i.e. all P patterns occur with equal frequency, the occurrence of patterns is uncorrelated in time and strategies' scores are updated only after all P patterns are used. Within such setting we can relatively easily obtain analytical results, with no need to resort to computer simulations. In so doing, we rely on the results of Refs. [37] and [38], which show that MG with random uniformly chosen memories is very close, although not identical, to the canonical MG. All important qualitative aspects are the same and the small quantitative difference diminishes when we approach the critical point.

In sequel, we shall denote the average of a function $f^\mu \in \mathbb{R}$ taken over all the information patterns μ as $\overline{f^\mu} \equiv \frac{1}{P} \sum_{\mu=1}^P f^\mu$. Next, the action prescribed by strategy $s = \pm 1$ of i -th agent from group g , provided the information pattern is μ , is denoted by $a_{g,i}^{\mu,s} \in \{-1, +1\}$. Then, the currently executed action of an agent reads

$$a_{g,i}^\mu(t) \equiv a_{g,i}^{\mu, s_{g,i}(t)} \quad (2)$$

where the random variable $s_{g,i}(t)$ is an agent's current choice of strategy. It will be specified in detail later on. The following combinations

$$\begin{aligned} \omega_{g,i}^\mu &\equiv \frac{a_{g,i}^{\mu,+1} + a_{g,i}^{\mu,-1}}{2} \\ \xi_{g,i}^\mu &\equiv \frac{a_{g,i}^{\mu,+1} - a_{g,i}^{\mu,-1}}{2} \end{aligned} \quad (3)$$

allow us to write the aggregated action of agents from group g as

$$A_g^\mu(t) \equiv \sum_{i=1}^{N_g} a_{g,i}^\mu(t) = \sum_{i=1}^{N_g} \omega_{g,i}^\mu + s_{g,i}(t) \xi_{g,i}^\mu . \quad (4)$$

Next, for the overall impact of all individual actions of all agents we have

$$B^\mu(t) \equiv \sum_{g=1}^G I_g A_g^\mu(t) . \quad (5)$$

We follow the batch variant of the MG where the difference of strategies' scores $q_{g,i}(t)$ evolves as

$$q_{g,i}(t+1) - q_{g,i}(t) = -\overline{\xi_{g,i}^\mu B^\mu(t)} \quad (6)$$

Note that in this definition the update of the strategies' scores of an agent contains just the variable $\xi_{g,i}^\mu = \pm 1$ indicating the chosen side. The alternative updating of the score would be proportional to the impact I_g of the agent in question. We shall discuss the differences between these two choices later.

Now, the current strategy choice $s_{g,i}(t)$ of an agent is governed by the stochastic rule

$$\text{Prob}\{s_{g,i}(t) = \pm 1\} = \frac{1}{2} \frac{e^{\pm \Gamma q_{g,i}(t)}}{\cosh(\Gamma q_{g,i}(t))} \quad (7)$$

where the parameter $\Gamma \in \mathbb{R}^+$ is usually named as the “learning rate”. Indeed, the higher the value of Γ is, the more an agent trust his current knowledge represented by the value of $q_{g,i}(t)$. Henceforth, we denote the mean value of any random variable $x = x(s_{g,i}(t))$ with respect to the probability distribution (7) by angular brackets, $\langle x \rangle$. Especially, for the mean value of $s_{g,i}(t)$ itself, we have

$$m_{g,i}(t) \equiv \langle s_{g,i}(t) \rangle = \tanh(\Gamma q_{g,i}(t)) \quad (8)$$

In what follows, we shall refer this value as magnetization in analogy to purely physical systems.

In the next section, these magnetizations are necessary to evaluate the performances of the agents in the MG. Optimally, we would calculate all $m_{g,i}(t)$ as functions of time and made our conclusions. To ease this program a bit, we utilize here the pioneering way the MG was firstly solved using the replica method [30, 31]. Thus, we search for the value of $m_{g,i}(t)$ only at the stationary point of the game's dynamics. Such stationary value $m_{g,i}$ is defined in this manner

$$m_{g,i} \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T m_{g,i}(t) \quad (9)$$

The same convention will be used also for other variables to denote their stationary value.

To find the stationary state, we observe that, for $\Gamma \rightarrow 0$, the evolution of the strategies' scores (6) implies the following dynamics of magnetizations (8)

$$m_{g,i}(t+1) - m_{g,i}(t) \simeq -\frac{1}{2N} \frac{\Gamma}{I_g} (1 - m_{g,i}^2(t)) \frac{\partial H}{\partial m_{g,i}} \quad (10)$$

with the Lyapunov function

$$\begin{aligned} H(\{m_{g,i}\}_{g,i}, \{a_{g,i}^{\mu,\pm 1}\}_{g,i}) &= \overline{\left(\sum_{g=1}^G I_g \langle A_g^\mu \rangle \right)^2} = \\ &= \frac{1}{4} \overline{\left(\sum_{g=1}^G I_g \left(\sum_{i=1}^{N_g} a_{g,i}^{\mu,+} (1 + m_{g,i}) + a_{g,i}^{\mu,-} (1 - m_{g,i}) \right) \right)^2} \quad (11) \end{aligned}$$

which plays the role of the Hamiltonian in subsequent calculations. Next, by calculating the arguments of the minimum of H we obtain the stationary values of $m_{g,i}$.

If we make the score updates proportional to the actual impact of an agent, which is the alternative already discussed nearby Eq. (6), the equation (10) would require a modification. However, a short calculation reveals that the modification consists only in missing I_g in the denominator on the right-hand side of Eq. (10). Most importantly, both definitions result in the same Lyapunov function and further course of the calculations is therefore identical. Thus, it does not matter whether the prescription for the dynamics of strategies' scores (6) is proportional to an actual impact or not. However, what does matter, is the definition of the outcome of the game for an agent, as defined later in Eq. (14). The choice (6) we did earlier was taken to make this definition conform with the definition (14) (to be seen later).

In fact, the previous approach with Hamiltonian H playing the role of Lyapunov function is not the most general. To determine the behaviour of the MG for arbitrary Γ , there exists another path of calculations mastering stochastic calculus [32]. However, it was shown that finite value of Γ influences the results only in the non-ergodic phase while in the ergodic phase the results are independent of Γ . Here we investigate only the ergodic phase, thus the limit $\Gamma \rightarrow 0$ is acceptable and the use of Lyapunov function is justified.

In the course of computations, we shall work in the thermodynamic limit of infinite number of agents, $N \rightarrow \infty$, keeping all group ratios λ_g fixed, and at the same time limiting the number of information patterns to infinity, $P \rightarrow \infty$, holding the fraction

$$\alpha \equiv \frac{P}{N} \quad (12)$$

fixed. This value is the principal parameter of both canonical and generalized MG.

3. The profit of agents

In this section, we have to detour from the conventional notation used in the context of the MG. To incorporate the utility function of agents, it is better to write about their profits instead of their losses. To establish the connection to the canonical MG, a clear relation to volatility

$$\sigma^2 \equiv \frac{1}{N} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \overline{B^\mu(t)^2}, \quad (13)$$

which is the usual characteristic of agents' performance, will be derived.

First of all, we define the current outcome of an individual agent i of group g as

$$\rho_{g,i}(t) \equiv -\overline{a_{g,i}^\mu(t) B^\mu(t)}. \quad (14)$$

It indicates whether an agent was successful in the current round of the batch MG. His ability to obey the “join-minority imperative”, i.e. to choose $a_{g,i}^\mu(t) = -\text{sign}(B^\mu(t))$ for a typical information pattern μ , guarantees him positive outcome $\rho_{g,i}(t)$ and the same holds vice versa. We note that the fact that an agent's impact I_g is not included in formula (14) is the crucial point of our modification of the MG.

Next, for the average outcome within the group g we have

$$\rho_g(t) \equiv \frac{1}{N_g} \sum_{i=1}^{N_g} \rho_{g,i}(t) = -\frac{1}{N_g} \overline{\langle A_g^\mu(t) B^\mu(t) \rangle} . \quad (15)$$

Coming to the stationary state, we can write (the stationary value of) the overall average outcome ρ as

$$\rho \equiv [\rho_g] \quad (16)$$

To relate an average outcome within a group to an average profit of that group, we have to take into account the utility function $U(I)$. We follow the standard way it is defined in classical economics. Its purpose is to quantify agents' preferences over the whole spectra of possible impacts [36]. In other words, the more an agent prefers impact I , the higher value is assigned to that impact by $U(I)$. Such utility function can be interpreted, for instance, as the profit of an owner of the vehicle of length I . For the sake of simplicity, we assume that the utility function is common for all agents of all groups and non-negative. Now, we define the average profit $\Pi_g^{\{U(I)\}}$ of group g provided the utility function is $U(I)$ in this manner

$$\Pi_g^{\{U(I)\}} \equiv U(I_g) \rho_g . \quad (17)$$

Then, the overall average profit over all groups reads

$$\Pi^{\{U(I)\}} \equiv \left[\Pi_g^{\{U(I)\}} \right] . \quad (18)$$

We note that introduction of $U(I)$ is by no means equivalent to readjusting of the impacts.

If we consider all impacts to have the same utility, i.e. $U(I) = 1$, we obtain an extremal case of profit $\Pi^{\{1\}} = \rho$. On the other hand, prescribing an utility of impact directly proportional to its size, i.e. $U(I) = I$, we observe $\Pi^{\{I\}} = -\sigma^2$, so the minus volatility (13) is another extremal case of the profit defined by formula (18). Realistic situations will lie somewhere in the middle. For example, we can choose $U(I) = I^d$ for $0 < d < 1$ or $U(I) = I/(I+1)$. However, in the following we shall mainly compare the extremal cases $-\sigma^2$ and ρ , to better see the difference. We shall see that this shift of focus has important implications on the conclusions we shall draw from the results of the computations.

At the moment, we define the mutual predictability between groups $f, g \in \{1, \dots, G\}$

$$\theta_{fg} \equiv \frac{N}{N_f N_g} \overline{\langle A_f^\mu \rangle \langle A_g^\mu \rangle} \quad (19)$$

which is a direct generalisation of the usual definition of the predictability in the canonical MG [9]. It corresponds to the ability of group f to adapt to the aggregated attendances A_g , and vice versa. The less the value of θ_{fg} is, the more the groups f, g are adopted to each other as their aggregated attendances A_f^μ and A_g^μ are more anti-correlated. Then, for the mean predictability of group g it holds

$$\theta_g \equiv \frac{1}{N} \overline{\langle A_g^\mu \rangle \langle B^\mu \rangle} = [I_f \theta_{fg}] . \quad (20)$$

Its knowledge is the key ingredient in the calculation of ρ_g . Indeed, if we apply the approximation

$$\sum_{f=1}^G I_f \sum_{i=1}^{N_g} \sum_{j=1}^{N_f} \overline{\xi_{g,i}^\mu \xi_{f,j}^\mu} \langle (s_{g,i} - m_{g,i})(s_{f,j} - m_{f,j}) \rangle \simeq I_g \sum_{i=1}^{N_g} \overline{(\xi_{g,i}^\mu)^2} (1 - m_{g,i}^2) \quad (21)$$

which is justified in the ergodic phase, see again Ref. [9], introduce the set of order parameters for $g \in \{1, \dots, G\}$

$$Q_g \equiv \frac{I_g^2}{N_g} \sum_{i=1}^{N_g} m_{g,i}^2, \quad (22)$$

and compare definitions of ρ_g and θ_g , see (15) and (20), respectively, we obtain

$$\rho_g = \frac{Q_g - I_g^2}{2I_g} - \theta_g. \quad (23)$$

At this moment, we can realise the importance of θ_g . It is not only a characteristic of groups' ability to predict the overall impact B^μ , but it is also a link connecting the knowledge of the outcome ρ_g of group g to the knowledge of the minima of Hamiltonian H . Recalling the definition of θ_{fg} , see (19), together with equation (11), we can derive that

$$\theta_{fg} = \frac{1}{2} \frac{N}{N_f N_g} \frac{d^2 H}{dI_f dI_g}. \quad (24)$$

Thus, as soon as we know H as function of the impacts I_g , $g \in \{1, \dots, G\}$, we deduce also all mean predictabilities θ_g for all groups using the mutual predictabilities θ_{fg} .

4. Replica solution

Whatever utility function $U(I)$ we choose for the impact of the agents, the key elements for the calculation are the average groups' outcomes ρ_g which are functions of the stationary magnetizations $m_{g,i}$, see (23). To obtain these magnetizations, it suffices to minimise the Hamiltonian $H(\{m_{g,i}\}_{g,i}, \{a_{\mu,g,i}^{\mu,\pm 1}\}_{g,i})$ as it is also Lyapunov function for their dynamics, remember equation (10). The technical obstacle to actual computation consists in the quenched disorder contained in the Hamiltonian (11) due to the random strategies $a_{g,i}^{\mu,\pm 1}$. The standard procedure to tackle this complication is the replica method [28, 29].

To briefly depict this method, consider a system with Hamiltonian $H_a(m)$ dependent on state variable m and also on random, but quenched disorder a distributed with probability $P(a)$. In that setting, even the partition function $Z_a = \int e^{-\beta H_a(m)} dm$ is a random variable and so is the free energy $F_a = -\frac{1}{\beta} \log Z_a$. Then, the physically relevant quantity is the mean free energy $F \equiv \int F_a P(a) da$. To ease its computation, the replicated partition function $Z(n)$ is introduced as the central notion of this method. It is calculated by replicating the system n -times with the same disorder a , and finally, by taking an average over this disorder

$$Z(n) \equiv \int Z_a^n P(a) da = \int e^{-\beta n F_a} P(a) da \quad (25)$$

As the second equality indicates, $Z(n)$ is also a characteristic function for the random variable F_a . Thus, for the mean free energy it holds

$$F = -\frac{1}{\beta} \lim_{n \rightarrow 0} \frac{dZ(n)}{dn} \quad (26)$$

However, this limiting process is only formal as we are unable to calculate $Z(n)$ for general n , except for positive integers. The existence and uniqueness of an analytical continuation of $Z(n)$ is an open question of the contemporary mathematical physics.

Full solution of the canonical MG using replica method is available [30, 31], as well as solution of various modifications of the MG [13, 14, 23, 25, 33, 34, 35]. We apply this method straightforwardly to our case. For the replicated partition function we obtain formula

$$Z(n) = \sum_{\{a_{g,j}^{\mu,s}\} \in \{-1,+1\}^{2PN}} \left(\int_{-1}^1 e^{-\beta H(\{m_{f,i}\}_{f,i}, \{a_{g,j}^{\mu,\pm 1}\}_{g,j})} \prod_{f,i} dm_{f,i} \right)^n \quad (27)$$

where the sum goes over all possible realizations of the quenched disorder. Using Hubbard-Stratonovich transformation, we introduce auxiliary variables $z^a \in \mathbb{R}$ indexed by replica index $a \in \{1, \dots, n\}$. Thus, we obtain the previous formula in the following form

$$Z(n) = \int_{-1}^1 \left(\int_{\mathcal{R}} e^{-\frac{1}{2} \sum_{a=1}^n (z^a)^2 - \frac{\beta}{2P} \sum_{a,b=1}^n z^a z^b \sum_{g,i} I_g^2 (1 + m_{g,i}^a m_{g,i}^b)} \prod_a dz^a \right)^P \prod_{a,g,i} dm_{g,i}^a \quad (28)$$

To compute it, it is necessary to introduce the replica-symmetric ansatz and then apply the saddle-point method. Next, we calculate the mean free energy F using equation (26). Finally, to achieve the formula for the minima of Hamiltonian H , it suffices to carry out the following limit

$$H = \lim_{\beta \rightarrow \infty} F \quad (29)$$

Without resorting more details of the computation, we note that at the core there is an equation for an auxiliary quantity $\zeta \in \mathbb{R}^+$ parameterized by α defined in (12)

$$\left[4\zeta^2 I_g^2 - 2\zeta^2 I_g^2 \operatorname{erf}(\zeta I_g) + \operatorname{erf}(\zeta I_g) - \frac{2\zeta I_g}{\sqrt{\pi}} e^{-\zeta^2 I_g^2} \right] = \alpha. \quad (30)$$

Once ζ is found, we can calculate the order parameters

$$Q_g = I_g^2 - I_g^2 \operatorname{erf}(\zeta I_g) + \frac{1}{2\zeta^2} \operatorname{erf}(\zeta I_g) - \frac{1}{\sqrt{\pi}} \frac{I_g}{\zeta} e^{-\zeta^2 I_g^2}, \quad (31)$$

the minimum value of the Hamiltonian

$$H = \frac{N}{2} [Q_g + I_g^2] \left(1 - \frac{[\operatorname{erf}(\zeta I_g)]}{\alpha} \right)^2 \quad (32)$$

and hence the mutual predictability using relation (24) reads

$$\begin{aligned} \theta_{fg} = & \frac{1}{\alpha} \left(2\zeta I_g - \zeta I_g \operatorname{erf}(\zeta I_g) - \frac{1}{\sqrt{\pi}} e^{-\zeta^2 I_g^2} \right) \times \\ & \times \left(2\zeta I_f - \zeta I_f \operatorname{erf}(\zeta I_f) - \frac{1}{\sqrt{\pi}} e^{-\zeta^2 I_f^2} \right) + \\ & \frac{\alpha - [\operatorname{erf}(\zeta I_h)]}{2\alpha} \left(\frac{2 - \operatorname{erf}(\zeta I_f)}{\lambda_f} \delta_{fg} - I_g I_f \frac{(2 - \operatorname{erf}(\zeta I_g))(2 - \operatorname{erf}(\zeta I_f))}{[2I_h^2 - I_h^2 \operatorname{erf}(\zeta I_h)]} \right). \end{aligned} \quad (33)$$

Considering equation (23), we conclude

$$\rho_g = \frac{\operatorname{erf}(\zeta I_g)}{4\zeta^2 I_g} + \frac{[\operatorname{erf}(\zeta I_f)]}{2\alpha\zeta} \left(2\zeta I_g - \zeta I_g \operatorname{erf}(\zeta I_g) - \frac{1}{\sqrt{\pi}} e^{-\zeta^2 I_g^2} \right) - I_g. \quad (34)$$

The expression (34) is the main result we shall build on further in the following sections. Before going to that, we complete the calculations by the formula for the volatility (13) which is

$$\sigma^2 = [I_g^2] + \frac{[\operatorname{erf}(\zeta I_g)]^2}{4\alpha\zeta^2} - \frac{[\operatorname{erf}(\zeta I_g)]}{2\zeta^2}. \quad (35)$$

In the following two sections, we shall analyse these results first at the critical point and then deep in the ergodic regime.

5. Critical behaviour

For the minimization of H the analysis of critical behaviour is crucial. The calculation of (27) is valid only in the ergodic phase of the MG as it was already examined in Ref. [9]. Therein, the formula for susceptibility χ is obtained which in our setting directly generalizes to

$$\chi = \frac{[\operatorname{erf}(\zeta I_g)]}{\alpha - [\operatorname{erf}(\zeta I_g)]} \quad (36)$$

And it is the very fact of diverging susceptibility what characterises the critical point. Thus, it was deduced that the ergodic phase of the MG corresponds to values $\alpha \geq \alpha_c$ with the critical value α_c obtained from

$$\alpha_c = [\operatorname{erf}(\zeta_c I_g)] \quad (37)$$

with ζ_c given by an equation emerging from the previous equation together with (30)

$$\left[2\zeta_c^2 I_g^2 - \zeta_c^2 I_g^2 \operatorname{erf}(\zeta_c I_g) - \frac{1}{\sqrt{\pi}} \zeta_c I_g e^{-\zeta_c^2 I_g^2} \right] = 0 \quad (38)$$

Considering only one group of players, critical coefficient $\alpha_c \simeq 0.3374$ of the canonical MG reappears. Generally, the critical value α_c is equal or below this “canonical” value for any game configuration $\{\lambda_g, I_g\}_g$ due to the concavity of $\operatorname{erf}(x)$ in (37).

In the canonical MG, the predictability vanishes at the critical point $\alpha = \alpha_c$. This observation does not hold for the mutual predictabilities θ_{fg} directly, however, it holds for

the mean predictability θ_g of an arbitrary group g . To verify it, we substitute condition (37) for α_c into the definition of θ_g , see (20), using equation (33). The achieved equation for θ_g at the critical point is directly proportional to the left side of (38) and on that account θ_g does vanish here.

Another significant characterization of the critical point in the canonical MG was related to the notion of frozen agents firstly mentioned in Ref. [39]. An agent i of group g is called frozen if one of his strategies is significantly better than the other. Thus, in the stationary point it holds $m_{g,i} = \pm 1$. In the canonical MG the fraction of frozen agents ϕ is maximized at the critical point and it decreases with increasing $\alpha > \alpha_c$. For the fraction of frozen agents ϕ_g within group g in the MG with heterogeneous impacts we obtain

$$\phi_g = 1 - \text{erf}(\zeta I_g) \quad (39)$$

following Ref. [40]. Then, the overall fraction of frozen agents is

$$\phi = 1 - [\text{erf}(\zeta I_g)] . \quad (40)$$

From equation (30) we can see that ζ grows with increasing α . Thus, even in the MG with heterogeneous impacts the overall ratio of frozen agents ϕ is maximized at $\alpha = \alpha_c$ and decreases with increasing $\alpha > \alpha_c$. Furthermore, this observation holds regardless of the actual game configuration $\{\lambda_g, I_g\}_g$.

Also, the loss in the canonical MG, which is represented by volatility $\sigma^2 = -\Pi^{\{I\}}$, was minimized at the critical point. We will show that this fact is violated in the generalized MG. Substituting the condition (37) for α_c into the formula for a group outcome ρ_g , remember (15), we obtain a simplified formula at the critical point

$$\rho_g = \frac{1}{2} \text{erf}(\zeta_c I_g) \left(\frac{1}{2 \zeta_c^2 I_g} - I_g \right) - \frac{1}{2 \zeta_c \sqrt{\pi}} e^{-\zeta_c^2 I_g^2} \quad (41)$$

We can show that $\rho_g \leq 0$ holds for all groups g and all the possible game configurations $\{\lambda_g, I_g\}_g$ as long as we are at the critical point. It implies that even the average profit $\Pi^{\{U(I)\}}$ is negative here for an arbitrary utility function $U(I)$ (which is non-negative by definition). Nevertheless, in the next section, we will see that there exist game configurations where $\Pi^{\{U(I)\}} > 0$ holds for some specific utility functions away from the critical point. The existence of positive average profits away from critical point implies that the critical point is no more optimal for the generalized variant of the MG. However, this observation depends on the considered utility function.

6. Two groups

Next, we examine the easiest generalisation of the canonical MG. We consider a game consisting of two groups of agents, $G = 2$, and depict their co-habitation. Firstly, we fix a scale of impacts I_g

$$[I_g^2] = 1. \quad (42)$$

This particular choice establishes a relation to the canonical MG as $[I_g^2]$ corresponds to volatility σ^2 in the game of randomly guessing agents. Thus, having this normalisation, we work on the scale of the canonical MG. We also prefer this normalisation because the

calculations for the game with infinitely many groups are valid only provided $[I_g^2] < \infty$ [26].

In all the subsequent figures it suffices to consider impact $I_1 \in [0, 1]$ since the rest of function values, i.e. those of argument $I_1 > 1$, are determined through (42) by reversing the groups. As for the variable λ_1 , we restrict ourselves to $\lambda_1 \in [0, 1]$. It is possible to prove that $\alpha_c(\lambda_1, I_1)$ is discontinuous along the line $\lambda_1 = 1$ considering the normalisation (42). In Fig. 1, it is clearly visible that the critical value $\alpha_c(\lambda_1, I_1)$ is, indeed, lower or equal to the critical value $\alpha_c \simeq 0.3374$ observed in the canonical MG. We picked $\alpha = 0.4$ for all subsequent calculations in order to remain safely in the ergodic phase as it was discussed in the previous section.

The principal novelty in our result is that for some game configurations $\{\lambda_g, I_g\}_g$ the overall average outcome $\rho = \Pi^{\{1\}}$ can even be positive as depicted in Fig. 2. In other words, the majority of agents wins in these cases. From Fig. 3 and Fig. 4 showing separate outcomes of each group ρ_1 and ρ_2 , respectively, we can deduce what is behind this phenomena. To gain on average, one group has to sponge off the other. In the case of two groups, the group with smaller impact is always winning. Its positive outcome is not very high, in fact, but when weighted by the groups' ratios, it can overweight negative outcome of the other group.

In the next three figures, we plot the predictabilities θ_{11} , θ_{22} and mutual predictability θ_{12} , see Fig. 5, Fig. 6 and Fig. 7, respectively. Note that the positiveness of the diagonal elements of θ_{fg} is enforced by their definition, see (19). Next, we observe that θ_{12} is negative for all configurations of the game. In other words, the aggregated actions A_1^μ and A_2^μ are anticorrelated for the MG consisting of two groups.

In Fig. 8, we show the volatility $\sigma^2 = -\Pi^{\{I\}}$ defined by (13). The closer σ^2 is to 1, the closer is the result of the game to the behaviour of randomly guessing agents. We observe that this holds in the vicinity of $\lambda_1 = 1$ and $I_1 = 0$. In other words, the co-adaptation of agents is poor for these game parameters. However, it is the same area where α_c falls almost to 0, confer Fig. 1. It resembles a similar feature of the canonical MG, where σ^2 grows with α receding from α_c . Here, the situation is analogous. Even though α was fixed in this set of calculations, α_c is decreasing in the region of our interest and thus the distance $\alpha - \alpha_c$ is growing.

The next three figures depict the fraction of frozen agents. We plot their overall fraction ϕ , fraction of frozen agents within the first group ϕ_1 and fraction of frozen agents within the second group ϕ_2 , see Fig. 9, Fig. 10 and Fig. 11, respectively. For higher ratio of frozen agents we observe also the higher overall outcome, compare ϕ with ρ . This was valid also in the canonical MG, where frozen agents were more effective on average.

In the last set of figures, we depict the way the overall average profit $\Pi^{\{U(I)\}}$ is influenced by the particular choice of the utility function $U(I)$. The overall average profits $\Pi^{\{I^{1/2}\}}$ and $\Pi^{\{I^{1/4}\}}$ are depicted in Fig. 12 and Fig. 13, respectively. For the utility function $U(I) = I^{1/4}$ we append Fig. 14 and Fig. 15 depicting a separate average profits of both groups $\Pi_1^{\{I^{1/4}\}}$ and $\Pi_2^{\{I^{1/4}\}}$, respectively. Next, we add another overall average profit $\Pi^{\{I/(I+1)\}}$ using a different shape of the utility function, see Fig. 16. For all these utility functions new local maxima of $\Pi^{\{U(I)\}}$ appear.

Although we did not solve explicitly the problem of maximizing the agents' utility, we consider the calculated dependence of the average profit on impacts as a guide for a

hypothetical choice of optimal impact. The location of the optimum depends crucially on the form of the utility function. Therefore, the introduction of the utility function is one of the cornerstones of our approach.

7. Conclusion

We investigated in detail the modification of the Minority Game in which the agents act on different scales. These heterogeneous scales are implemented as agents' impacts and can be interpreted in different ways. Within the traffic metaphor of the MG, the impacts corresponds to the lengths of the vehicles the agents are driving. Alternatively, we can view it as a simplified method how to account for different frequencies at which the agents are playing. Then, an agent's impact is directly proportional to the inverse of the time scale at which the agent is participating.

The agents are divided into several groups according to their impacts. The MG model with heterogeneous impacts is solved by standard procedure using the replica trick. The average outcome of each group was then obtained, as well as the mutual predictabilities of the groups. To interpret properly the ensemble of outcomes for each group, we introduce an utility function of the impact of agents. Within the traffic metaphor, utility says how much profit can be expected having vehicle of the given length. The average profit of individual groups is calculated from their average outcomes, with utility considered. We discuss the overall average profit of the system for the simplest case of two groups. The agents with lower impact are always in advantage, and the overall profit of the system can even be positive, which means that the MG viewed from this perspective may be a positive-sum game. This is in contrast with the usual way of measuring the losses through the volatility which enforces that the MG is a negative-sum game.

Although we kept the traffic metaphor of the MG within this paper, we believe that the results can be interpreted also in the language of stock markets. We argue that if we think of the impact of an agent as the size of the capital an investor is playing with, the relative outcome ρ_g is more suitable description of an agent's profit. The real investors do compete for highest relative rather than absolute profit. We see that in absolute numbers the MG with heterogeneous impacts remains, of course, a negative-sum game, but subjectively felt relative profit of agents averaged over the whole system may be positive. This observation could give a cause for the heterogeneity of investors' sizes, which is one of the most fundamental features of the stock market [27]. Interpreting the utility function $U(I)$ as an investor's subjective evaluation of the capital unit I , the notion of $\Pi^{\{U(I)\}}$ is a tool suitable for finer analysis of this phenomena.

To sum up, by introduction diversity in the impact of agents, the average profit in the MG, measured by an non-negative utility function, may turn the MG into a positive-sum game, so that the diversity implies advantage for all. We can conjecture that this is the driving mechanism which favours emergence of groups acting with different impacts. An important question which still remains to be solved is, what would happen if each of the agents is free in her choice of the impact. Both group sizes and group impacts would be free to be adjusted to an optimum. The investigation of the properties of these Nash equilibria is left to a future publication.

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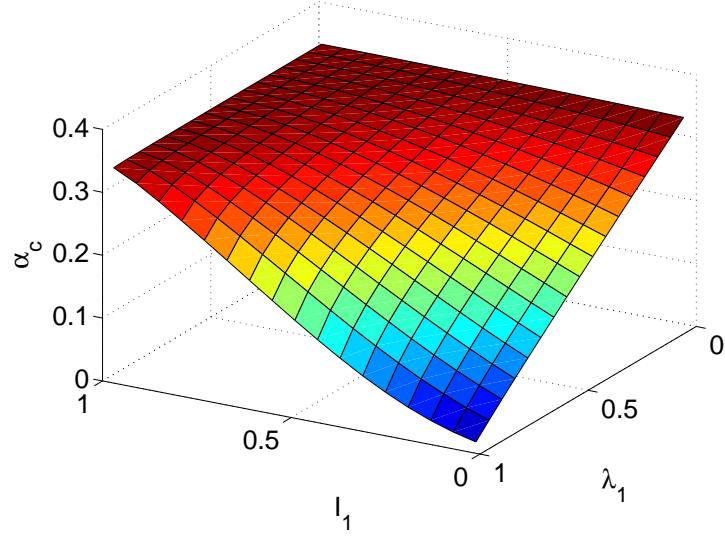


Figure 1: Critical alpha $\alpha_c(\lambda_1, I_1)$ calculated for the game of two groups ($G = 2$) with groups' impacts normalized as $[I_g^2] = 1$.

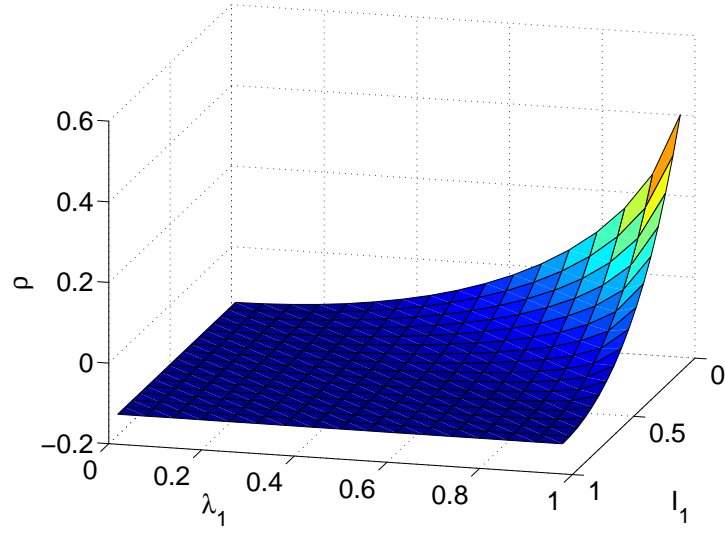


Figure 2: The average overall outcome $\rho(\lambda_1, I_1)$ calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

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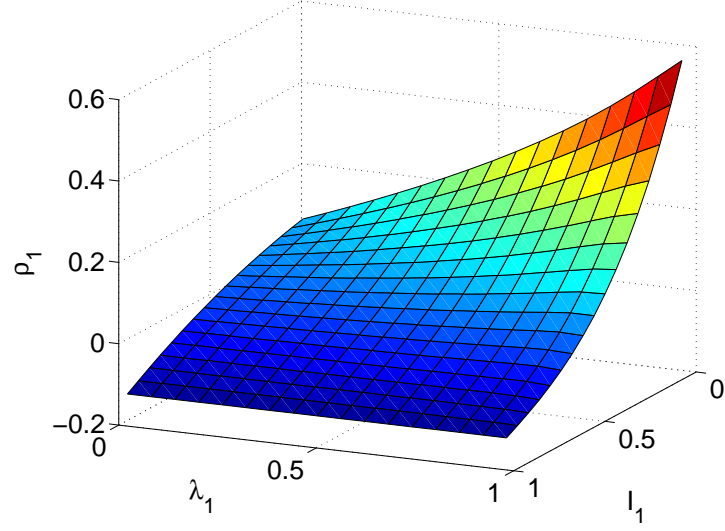


Figure 3: The average outcome $\rho_1(\lambda_1, I_1)$ of the first group calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

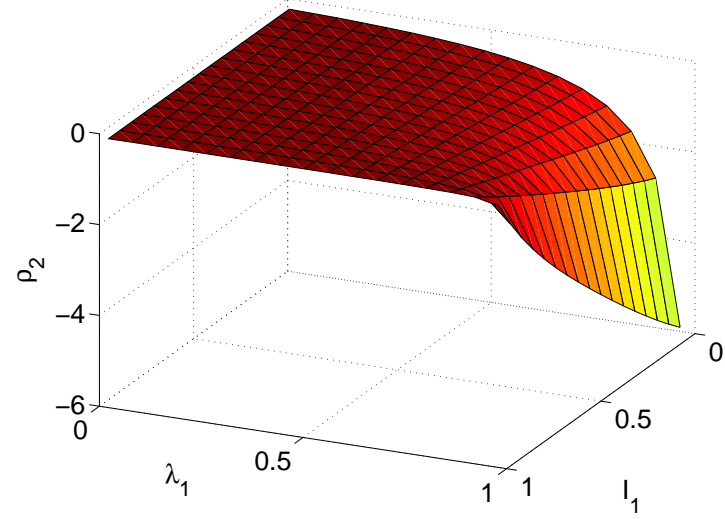


Figure 4: The average outcome $\rho_2(\lambda_1, I_1)$ of the second group calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

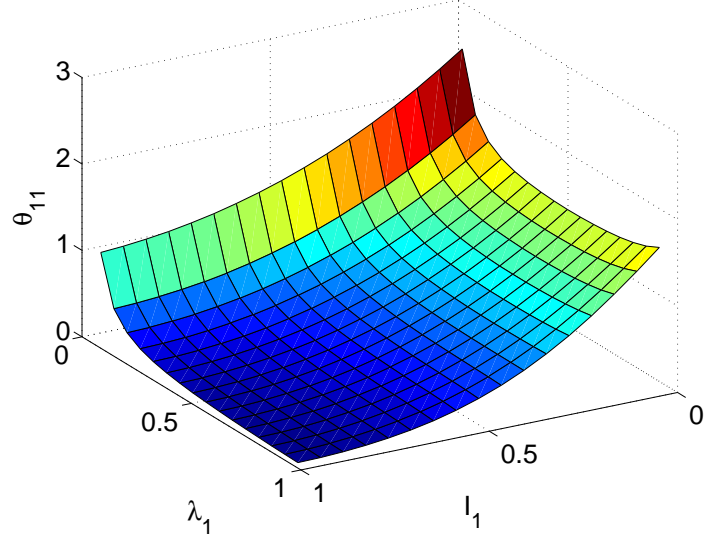


Figure 5: The predictability $\theta_{11}(\lambda_1, I_1)$ of the first group calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

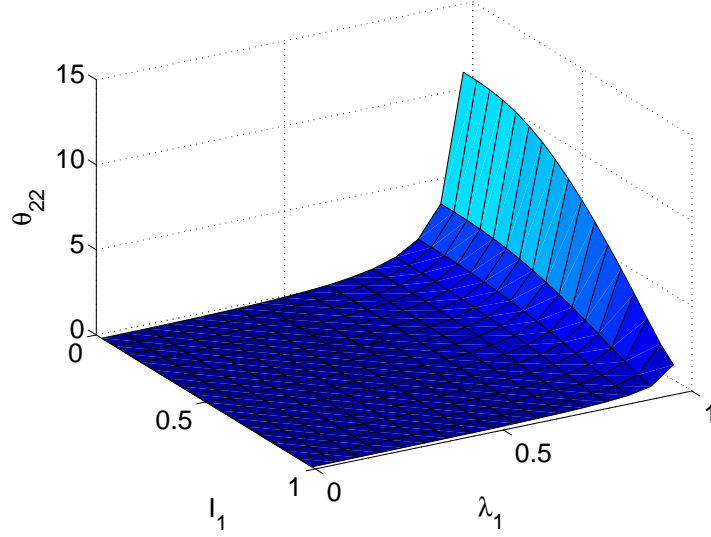


Figure 6: The predictability $\theta_{22}(\lambda_1, I_1)$ of the second group calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

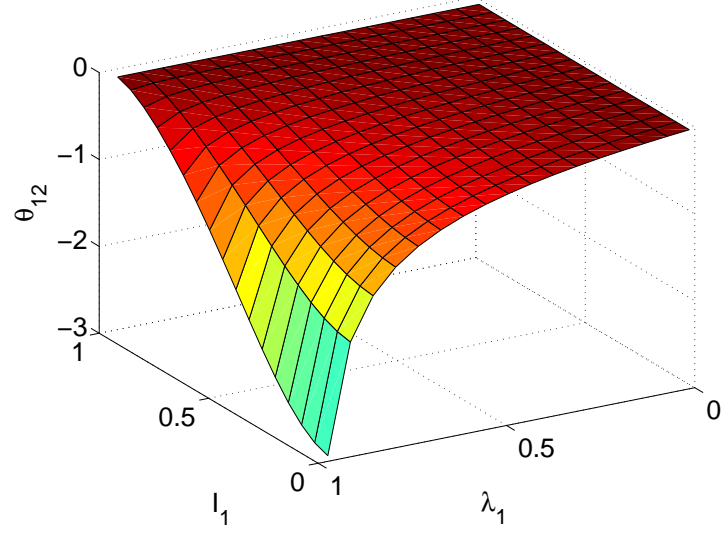


Figure 7: The mutual predictability $\theta_{12}(\lambda_1, I_1)$ between groups calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

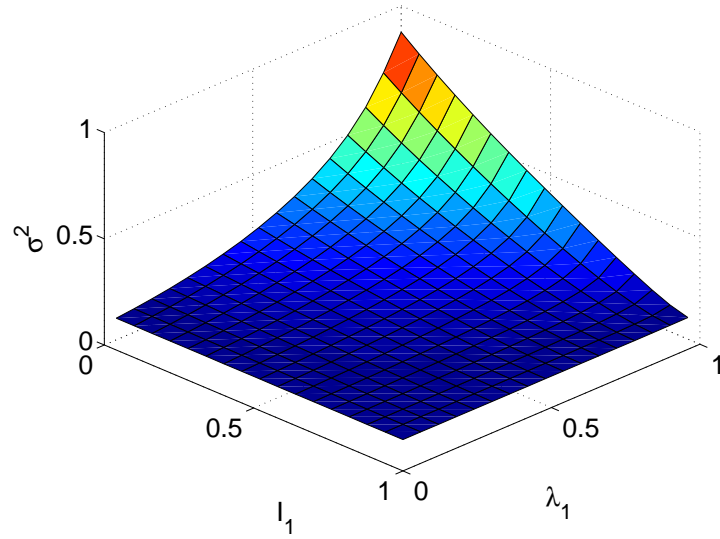


Figure 8: The volatility $\sigma^2(\lambda_1, I_1)$ calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

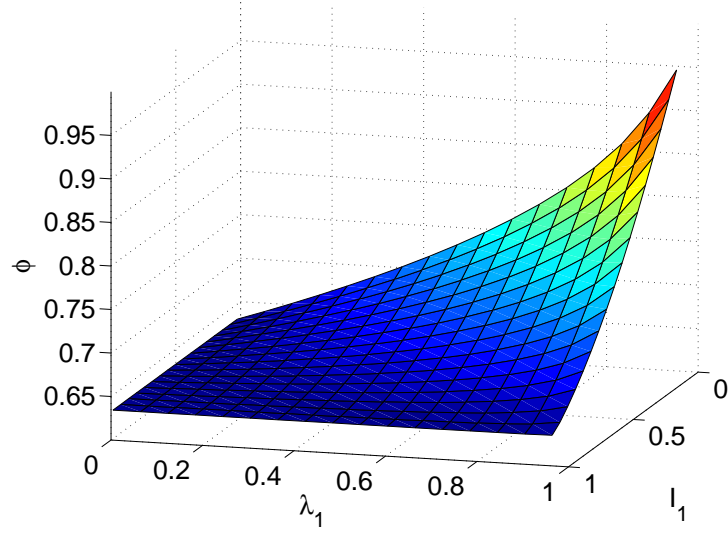


Figure 9: The average ratio of frozen agents $\phi(\lambda_1, I_1)$ calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

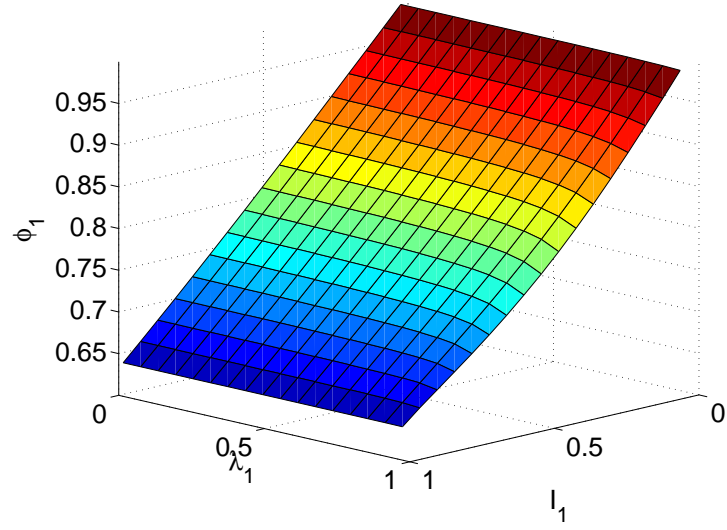


Figure 10: The ratio of frozen agents $\phi_1(\lambda_1, I_1)$ within the first group calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

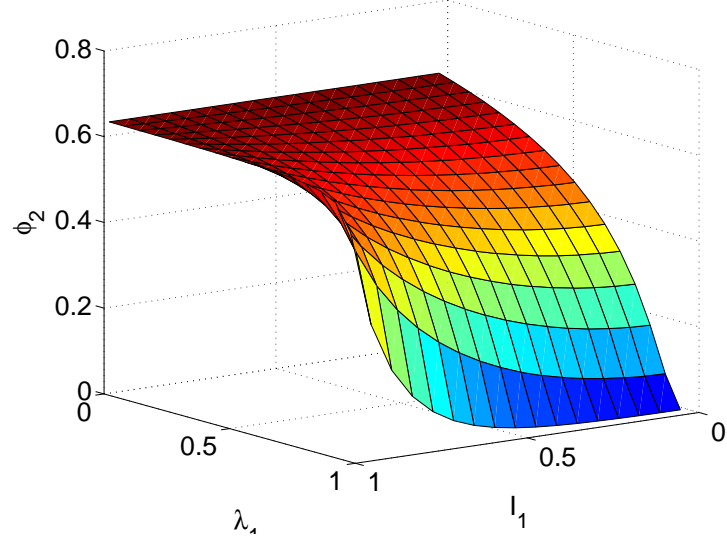


Figure 11: The ratio of frozen agents $\phi_2(\lambda_1, I_1)$ within the second group calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

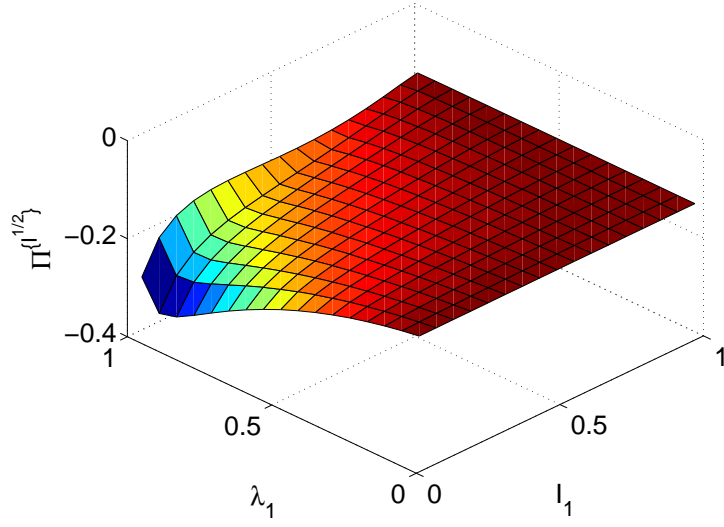


Figure 12: The average overall profit $\Pi^{\{I^{1/2}\}}(\lambda_1, I_1)$ calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

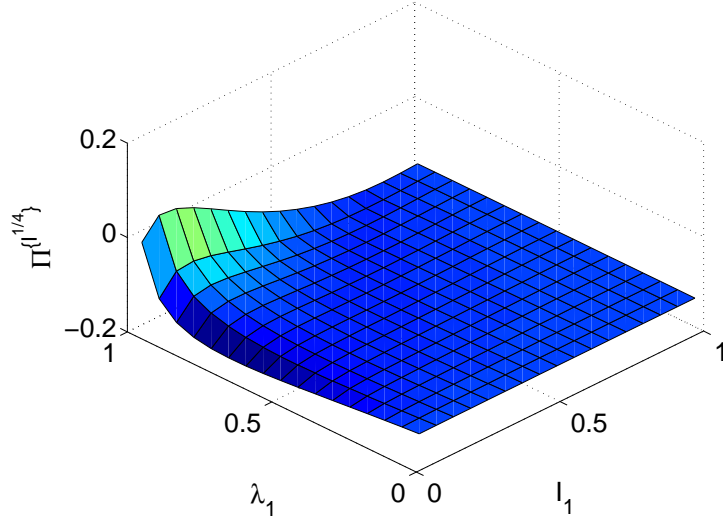


Figure 13: The average overall profit $\Pi^{\{I^{1/4}\}}(\lambda_1, I_1)$ calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

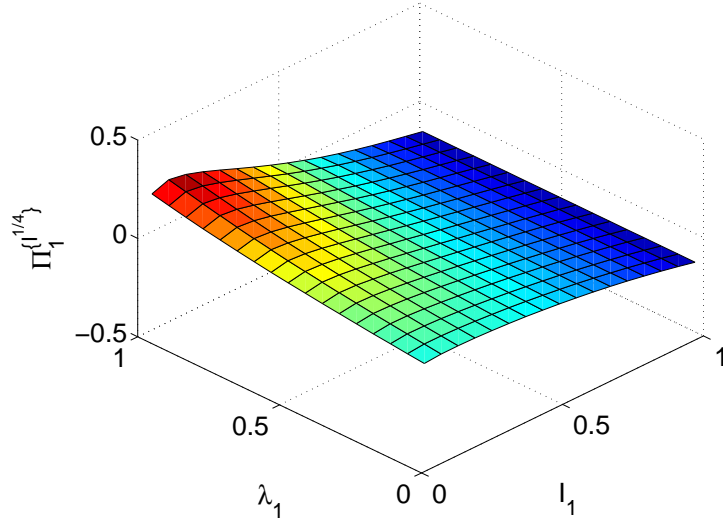


Figure 14: The average profit $\Pi_1^{\{I^{1/4}\}}(\lambda_1, I_1)$ of the first group calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

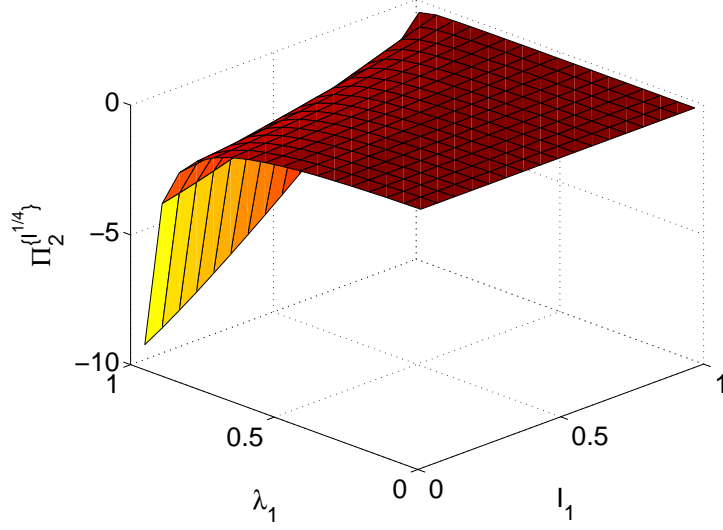


Figure 15: The average profit $\Pi_2^{\{I^{1/4}\}}(\lambda_1, I_1)$ of the second group calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

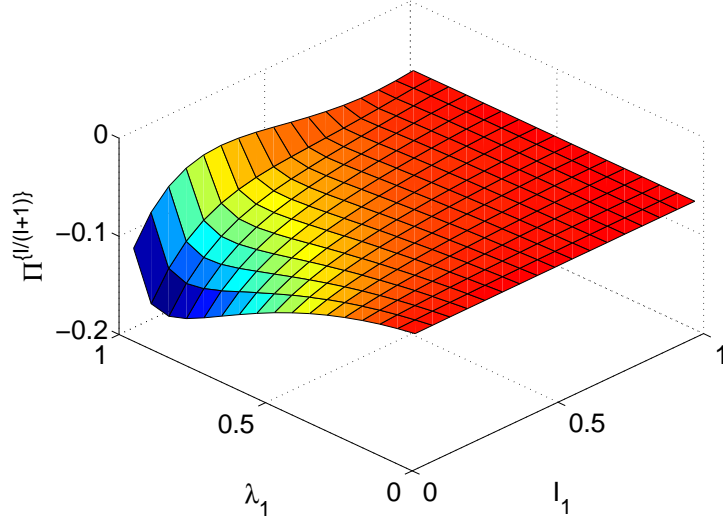


Figure 16: The average overall profit $\Pi^{\{I/(I+1)\}}(\lambda_1, I_1)$ calculated for the game of two groups ($G = 2$) with $\alpha = 0.4$ and groups' impacts normalized as $[I_g^2] = 1$.

